ON A THEOREM OF HARDY AND RAMANUJAN

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Let $m = p_1^{a_1} \ldots p_r^{a_r}$, $f(m) = a_1 + \ldots + a_r$, $V(m) = r$.

Hardy and Ramanujan have proved the following theorem[†]:

If $\delta > 0$, $A > 0$, then the number of integers n not exceeding N, for which one of the inequalities

$$(1) \qquad \begin{cases} V(n) > \log\log N + A(\log\log N)^{\frac{1}{2}+\delta}, \\ V(n) < \log\log N - A(\log\log N)^{\frac{1}{2}+\delta}, \end{cases}$$

holds, is $o(N)$; and the same is true for $f(n)$ also.

I give here a simple and elementary proof which differs entirely from that due to Hardy and Ramanujan.

Let

$$(2) \qquad R(N) = \sum_{n=1}^{N} \{V(n) - \log\log N\}^2.$$

If one of the inequalities (1) was true for $\Omega(N)$ numbers below N, then $R(N)$ would be evidently $\Omega\{N(\log\log N)^{1+2\delta}\}$. We shall, however, prove that

$$(3) \qquad R(N) = O(N \log\log N),$$

and this will prove the theorem for $V(n)$. Similarly I shall show that

$$D(N) = \sum_{n=1}^{N} \{f(n) - \log\log N\}^2 = O(N \log\log N),$$

and the theorem for $f(n)$ will follow similarly.

LEMMA 1.

$$\sum_{n=1}^{N} \{V(n)\}^2 = \sum_{i \neq h} \left[\frac{N}{p_i p_h}\right] + \sum_{i} \left[\frac{N}{p_i}\right].$$

For $\{V(n)\}^2 = \left(\sum_{p_i | n} 1\right)^2 = \sum 1$, taken as many times as it is possible to choose different pairs (p_i, p_h) of prime factors of n; for this purpose (p_i, p_h) and

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[†] *Quart. J. of Math.*, 48 (1920), 76–92 (77).

(p_h, p_i) are reckoned as different pairs. Evidently the contribution of an arbitrary pair (p_i, p_h) to $\sum_1^N V(n)^2$ is equal to $[N/(p_i p_h)]$ for $i \neq h$, and to $[N/p_i]$ for $i = h$. This establishes our lemma.

In the same way we may prove

LEMMA 2.

$$\sum_{n=1}^{N} \{f(n)\}^2 = \sum_{i \neq h, \alpha, \beta} \left[\frac{N}{p_i^{\alpha} p_h^{\beta}}\right] + \sum_{i, \alpha} \left[\frac{N}{p_i^{\alpha}}\right] (2\alpha - 1).$$

Further, we have, obviously,

LEMMA 3.

$$\sum_{n=1}^{N} V(n) = \sum_{i} \left[\frac{N}{p_i}\right], \qquad \sum_{n=1}^{N} f(n) = \sum_{i, \alpha} \left[\frac{N}{p_i^{\alpha}}\right].$$

Finally, we prove

LEMMA 4.

$$A(N) \equiv \sum_{p_i p_h < N} \frac{1}{p_i p_h} = (\log \log N)^2 + O(\log \log N).$$

For we have

$$\Big(\sum_{p_i \leqslant \sqrt{N}} \frac{1}{p_i}\Big)^2 < \sum_{p_i p_h < N} \frac{1}{p_i p_h} < \Big(\sum_{p_i \leqslant N} \frac{1}{p_i}\Big)^2,$$

and, in virtue of the elementary relation

$$\sum_{p_i \leqslant M} \frac{1}{p_i} = \log \log M + O(1), \tag{4}$$

the lemma is proved.

Coming now to the proof of our main theorem, we have, in virtue of Lemmas 1 and 4,

$$\begin{aligned} \sum_{n=1}^{N} V(n)^2 &= N \sum_{p_i p_h < N} \frac{1}{p_i p_h} + O(N) + N \sum_{p_i \leqslant N} \frac{1}{p_i} + o(N) \\ &= N (\log \log N)^2 + O(N \log \log N). \end{aligned}$$

In consequence of Lemma 3,

$$\sum_{n=1}^{N} V(n) = N \sum_{p_i \leqslant N} \frac{1}{p_i} + o(N) = N \log \log N + o(N),$$

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and therefore

$$R(N) = \sum_{n=1}^{N} \{V(n) - \log\log N\}^2$$

$$= \sum_{1}^{N} V(n)^2 - 2\log\log N \sum_{1}^{N} V(n) + N(\log\log N)^2$$

$$= O(N \log\log N).$$

The inequality for $D(N)$ is proved in the same way.

If, instead of (4), we use the more precise but also elementary relation

$$\sum_{p_i \leqslant M} \frac{1}{p_i} = \log\log M + a + o(1),$$

where a is a constant, we obtain in the same way the results

$$R(N) = N \log\log N + o(N \log\log N)$$

and $$D(N) = N \log\log N + o(N \log\log N).$$

NOTE ON THE MAGNITUDE OF THE DIFFERENCE BETWEEN SUCCESSIVE PRIMES

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In his Cambridge Tract[†], Ingham states: "The prime number theorem shows that the average interval $p_{n+1} - p_n$ between (large) consecutive primes is about $\log p_n$, but there may be wide deviations in either direction from this average. There are strong indications on the one hand that the interval reduces infinitely often to the value 2 ...; but this has not been proved. The opposite problem, that of abnormally large values of $p_{n+1} - p_n$, is also unsolved, and such indications as do exist are of a negative character. Thus, on the Riemann hypothesis, we infer easily from the results of this tract that, if ϑ is any fixed number greater than $\frac{1}{2}$, the interval is never as large as p_n^{ϑ}, except possibly in a finite number of instances, and it has been conjectured that the stronger assertion with $\vartheta = \frac{1}{2}$ is also true; but the most that has actually been proved is the corresponding statement with $\vartheta > 1 - (33000)^{-1}$[‡]".

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[*] Received and read 14 June, 1934.

[†] A. E. Ingham, "The distribution of prime numbers" (*Cambridge Tracts in Mathematics*, No. 30, 1932), 8.

[‡] G. Hoheisel, *Sitzungsberichte d. Preuss. Akad. d. Wiss., phys.-math. Kl.* (1930), 580.

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